perceive the world as divided into objects situated within space and time, not necessarily because it has this structure but because that is the only way we *could* perceive it.

Just because our brains navigate the world successfully does not mean they capture its structure faithfully. In machine learning, researchers have found that computer systems are often better at making predictions or controlling equipment when they eschew direct representations of the world. Similarly, reality might be completely unlike what our minds or our theories present to us. Scholars such as philosopher Colin McGinn and Harvard University psychologist Steven Pinker have suggested that our particular style of reasoning is why we find consciousness so hard. Perhaps one day we will construct artificial minds that see right through the problems that stump us, although they might get hung up on those we think are easy.

If anything restores confidence that truth is within our grasp, it is that we can divide and conquer. Although "real" is sometimes equated with "fundamental," each of the multiple levels of description in science has an equal claim to be considered real. Therefore, even if things vanish at the roots of nature, we are perfectly entitled to think of things in daily life. Even if quantum mechanics is mystifying, we can build a solid understanding of the world on it. And even if we worry that we aren't experiencing the fundamental reality, we are still experiencing *our* reality, and there's plenty to study there.

If we find that our theories are clutching at vapors, that's not a bad thing. It's reminding us to be humble. Physicists can be full of themselves, but the most experienced and accomplished among them are usually circumspect. They tend to be the first people to point out the problems with their own ideas, if only to avoid the embarrassment of someone else doing it for them. No one ever said that finding the truth would be easy. SA


**MORE TO EXPLORE**

**A Fight for the Soul of Science.** Natalie Wolchover in *Quanta Magazine.* Published online December 16, 2015. www.quantamagazine.org/physicists-and-philosophers-debate-the-boundaries-of-science-20151216

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**In It for the Long Haul.** Matthew Chalmers in *CERN Courier,* Vol. 59, No. 2, pages 45–49; March/April 2019. https://cerncourier.com/in-it-for-the-long-haul

**FROM OUR ARCHIVES**

Parallel Universes. Max Tegmark; May 2003.

Quantum Weirdness? It's All in Your Mind. Hans Christian von Baeyer; June 2013.

What Is Real? Meinard Kuhlmann; August 2013.

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MATHEMATICS

# NUMBERS GAME

## PHILOSOPHERS CANNOT AGREE ON WHETHER MATHEMATICAL OBJECTS EXIST OR ARE PURE FICTIONS

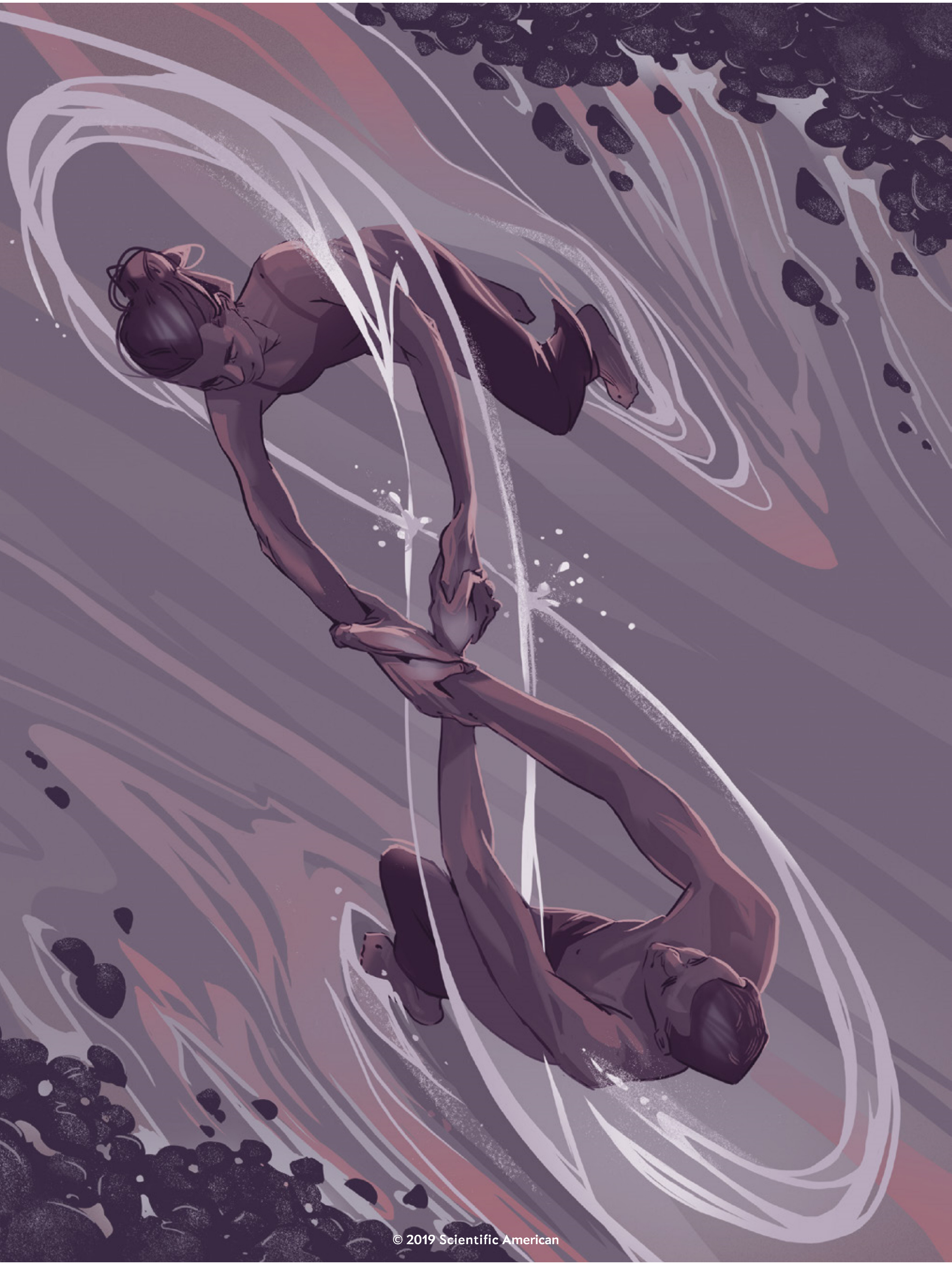
*By Kelsey Houston-Edwards*

[redacted]

**When I tell someone I am a mathematician, one of the most curious common reactions is: "I really liked math class because everything was either right or wrong. There is no ambiguity or doubt." I always stutter in response. Math does not have a reputation for being everyone's favorite subject, and I hesitate to temper anyone's enthusiasm. But math is full of uncertainties—it just hides them well.**

Of course, I understand the point. If your teacher asks whether 7 is a prime number, the answer is definitively "yes." By definition, a prime number is a whole number greater than 1 that is only divisible by itself and 1, such as 2, 3, 5, 7, 11, 13, and so on. Any math teacher, anywhere in the world, anytime in the past several thousand years, will mark you correct for stating that 7 is prime and incorrect for stating that 7 is not prime. Few other disciplines can achieve such incredible consensus. But if you ask 100 mathematicians what explains the truth of a mathematical statement, you will get 100 different answers. The number 7 might really exist as an abstract object, with primality being a feature of that object. Or it could be part of an elaborate game that mathematicians devised. In other words, mathematicians agree to a remarkable degree on whether a statement is true or false, but they cannot agree on what exactly the statement is about.

One aspect of the controversy is the simple philosophical question: Was mathematics discovered by humans, or did we invent it? Perhaps 7 is an actual object, existing independently of us, and mathematicians are discovering facts about it. Or it might be a figment of our imaginations



whose definition and properties are flexible. The act of doing mathematics actually encourages a kind of dual philosophical perspective, where math is treated as both invented and discovered.

This all seems to me a bit like improv theater. Mathematicians invent a setting with a handful of characters, or objects, as well as a few rules of interaction, and watch how the plot unfolds. The actors rapidly develop surprising personalities and relationships, entirely independent of the ones mathematicians intended. Regardless of who directs the play, however, the denouement is always the same. Even in a chaotic system, where the endings can vary wildly, the same initial conditions will always lead to the same end point. It is this inevitability that gives the discipline of math such notable cohesion. Hidden in the wings are difficult questions about the fundamental nature of mathematical objects and the acquisition of mathematical knowledge.

## INVENTION

HOW DO WE KNOW whether a mathematical statement is correct or not? In contrast to scientists, who usually try to infer the basic principles of nature from observations, mathematicians start with a collection of objects and rules and then rigorously demonstrate their consequences. The result of this deductive process is called a proof, which often builds from simpler facts to a more complex fact. At first glance, proofs seem to be key to the incredible consensus among mathematicians.

But proofs confer only conditional truth, with the truth of the conclusion depending on the truth of the assumptions. This is the problem with the common idea that consensus among mathematicians results from the proof-based structure of arguments. Proofs have core assumptions on which everything else hinges—and many of the philosophically fraught questions about mathematical truth and reality are actually about this starting point. Which raises the question: Where do these foundational objects and ideas come from?

Often the imperative is usefulness. We need numbers, for example, so that we can count (heads of cattle, say) and geometric objects such as rectangles to measure, for example, the areas of fields. Sometimes the reason is aesthetic—how interesting or appealing is the story that results? Altering the initial assumptions will sometimes unlock expansive structures and theories, while precluding others. For example, we could invent a new system of arithmetic where, by fiat, a negative number times a negative number is negative (easing the frustrated explanations of math teachers), but then many of the other, intuitive and desirable properties of the number line would disappear. Mathematicians judge foundational objects (such as negative numbers) and their properties (such as the result of multiplying them together) within the context of a larger, consistent mathematical landscape. Before proving a new theorem, therefore, a mathematician needs to watch the play unfold. Only then can the theorist know what to prove: the inevitable, unvarying conclusion. This gives the process of doing mathematics three stages: invention, discovery and proof.

The characters in the play are almost always constructed out of simpler objects. For example, a circle is defined as all points equidistant from a central point. So its definition relies on the definition of a point, which is a simpler type of object, and the distance between two points, which is a property of those simpler objects. Similarly, multiplication is repeated addition, and exponentiation is repeated multiplication of a number by itself. In consequence, the properties of exponentiation are inherited from the properties of multiplication. Conversely, we can learn about complicated mathematical objects by studying the simpler objects they are defined in terms of. This has led some mathematicians and philosophers to envision math as an inverted pyramid, with many complicated objects and ideas deduced from a narrow base of simple concepts.

In the late 19th and early 20th centuries a group of mathematicians and philosophers began to wonder what holds up this heavy pyramid of mathematics. They worried feverishly that math has no foundations—that nothing was grounding the truth of facts like $1 + 1 = 2$. (An obsessive set of characters, several of them struggled with mental illness.) After 50 years of turmoil, the expansive project failed to produce a single, unifying answer that satisfied all the original goals, but it spawned various new branches of mathematics and philosophy.

Some mathematicians hoped to solve the foundational crisis by producing a relatively simple collection of axioms from which all mathematical truths can be derived. The 1930s work of mathematician Kurt Gödel, however, is often interpreted as demonstrating that such a reduction to axioms is impossible. First, Gödel showed that any reasonable candidate system of axioms will be incomplete: mathematical statements exist that the system can neither prove nor disprove. But the most devastating blow came in Gödel's second theorem about the incompleteness of mathematics. Any foundational system of axioms should be consistent—meaning, free of statements that can be both proved and disproved. (Math would be much less satisfying if we could prove that 7 is prime and 7 is not prime.) Moreover, the system should be able to prove—to mathematically guarantee—its own consistency. Gödel's second theorem states that this is impossible.

The quest to find the foundations of mathematics did lead to the incredible discovery of a system of basic axioms, known as Zermelo-Fraenkel set theory, from which one can derive most of the interesting and relevant mathematics. Based on sets, or collections of objects, these axioms are not the idealized foundation that some historical mathematicians and philosophers had hoped for, but they are remarkably simple and do undergird the bulk of mathematics.

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IN BRIEF

**Mathematicians** tend to hold two simultaneous and incompatible views of the objects they study.

**Prime numbers,** for example, have surprising relations with one another that mathematicians are still discovering.

**Such explorations,** of what appears to be an alien landscape, encourage the idea that mathematical objects exist independently of humans.

**If mathematical objects** are real, however, why can one not touch, see or otherwise interact with them? Such questions often lead mathematicians to postulate that, in fact, the world of mathematical objects is fictitious.

*Illustration by Brook VanDevelder*